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# Unknotted Surfaces in 4-Space

By

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In this note we will discuss a concept of unknotted surfaces in the euclidean 4-space  $R^4$  and study elementary topics related to it. Spaces and maps will be considered from a piecewise-linear point of view. We will denote by  $R^3[t_0]$  the hyperplane whose fourth coordinate  $t$  is  $t_0$  in  $R^4$ , and for a subset  $A$  of  $R^3[0]$ ,  $A \times [a \leq t \leq b]$  means the subset  $\{(x, t) \in R^4 \mid (x, 0) \in A, a \leq t \leq b\}$  of  $R^4$ . The configurations of surfaces in  $R^4$  will be described by adopting the motion picture method. (cf. R.H.Fox[1], F.Hosokawa[4] or A.Kawauchi-T.Shibuya[6].)

## 1. A Concept of Unknotted Surfaces

Consider a closed, connected and oriented surface  $F_n$  of genus  $n$  ( $n \geq 0$ ) in  $R^4$ . We will assume that  $F_n$  is locally flat in  $R^4$ . It is reasonable to note the following known basic fact before stating our definition of unknotted surfaces: The surface  $F_n$  always bounds a compact, connected orientable 3-manifold in  $R^4$ .

[For example, to see this, consider the regular neighborhood  $N(F_n)$  of  $F_n$  in  $R^4$ . Since  $F_n$  is locally flat, we have  $N(F_n) = F_n \times D^2$  for a 2-cell  $D^2$ . The projection  $f: \partial N(F_n) (= F_n \times \partial D^2) \rightarrow \partial D^2$  is easily extendable to a piecewise-linear map  $\bar{f}: \text{cl}(R^4 - N(F_n)) \rightarrow \partial D^2$  by an elementary obstruction theory. Then the transverse-regularity argument assures us to find a compact, connected orientable 3-manifold  $M$  in  $\text{cl}(R^4 - N(F_n))$  with  $\partial M = F_n \times x$  for some  $x \in \partial D^2$ . This  $M$  may be extended to a manifold  $\bar{M}$  with  $\partial \bar{M} = F_n$  in  $R^4$ . See H. Gluck[2] or A. Kawauchi-T. Shibuya[6, Chapter II] for other more constructive proofs.] We will define an unknotted surface as the boundary of a solid torus in  $R^4$ . Precisely,

1.1. Definition.  $F_n$  is said to be unknotted in  $R^4$ , if there exists a solid torus  $T_n$  of genus  $n$  in  $R^4$  whose boundary  $\partial T_n$  is  $F_n$ . If such a  $T_n$  does not exist, then  $F_n$  is said to be knotted in  $R^4$ .

In the case of 2-spheres (i.e., surfaces of genera 0), Definition 1 is the usual definition of unknotted 2-spheres in  $R^4$  and it is well-known that any unknotted 2-sphere is ambient isotopic<sup>1)</sup> to the boundary of a 3-cell in the hyperplane  $R^3[0]$ .

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1) An ambient isotopy of a space  $X$  is a family  $\{h_t\}$  ( $0 \leq t \leq 1$ ) of auto-homeomorphisms of  $X$  with identity map  $h_0$ . For two subspaces  $X_1$  and  $X_2$  in a space  $X$ ,  $X_1$  is ambient isotopic to  $X_2$ , if there exists an ambient isotopy  $\{h_t\}$  of  $X$  with  $h_1(X_1) = X_2$ . An auto-homeomorphism  $f$  of  $X$  is ambient isotopic to the identity, if there exists an ambient isotopy  $\{h_t\}$  of  $X$  with  $h_1 = f$ .

The following theorem seems to justify Definition 1 for arbitrary unknotted surfaces.

1.2.Theorem.  $F_n$  is unknotted if and only if  $F_n$  is ambient isotopic to the boundary of a regular neighborhood of an  $n$ -leafed rose  $L_n$  in  $R^3[0]$ .

A 0-leafed rose  $L_0$  in  $R^3$  is understood as a point in  $R^3$ . For  $n \geq 1$  an  $n$ -leafed rose  $L_n$  in  $R^3$  is the union  $\bigcup_{i=1}^n \partial\Delta_i$  of the boundaries  $\partial\Delta_i$  of 2-simplices  $\Delta_i$  in  $R^3$  whose intersection  $\bigcap_{i=1}^n \Delta_i$  is one vertex of each  $\Delta_i$  and such that for each  $k, j, k \neq j$ ,  $\Delta_k \cap \Delta_j = \bigcap_{i=1}^n \Delta_i$ . In Fig. 1 below, we illustrated  $L_n$  for the case  $n = 6$ .

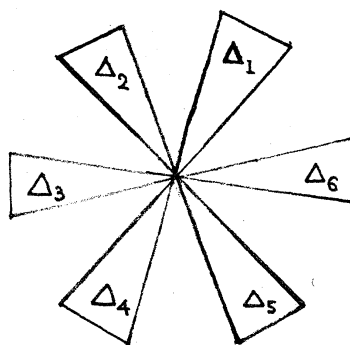


Fig. 1

1.3.Example. The surface of genus 1 in Fig. 2 is unknotted, since it bounds a solid torus of genus 1 that is shown in Fig.3.

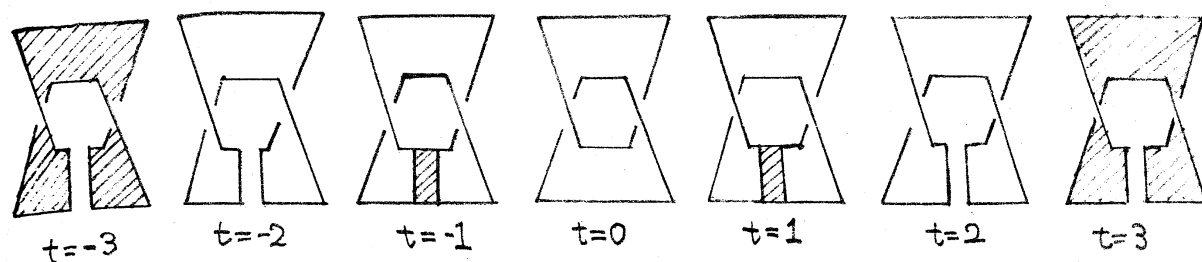


Fig. 2

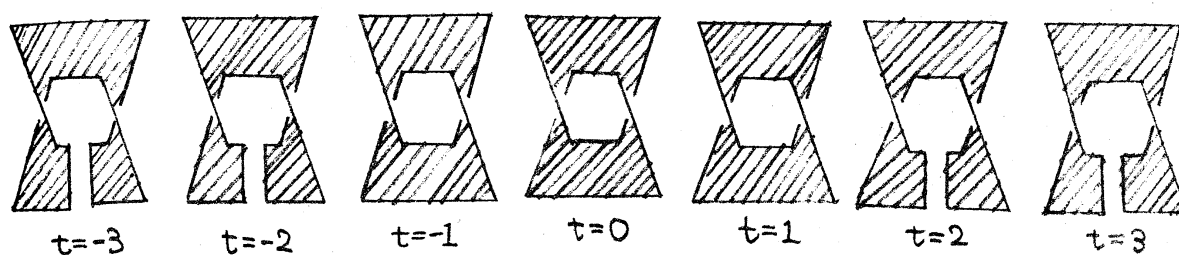


Fig. 3

Theorem 1.2 shows that this surface is ambient isotopic to the surface described in Fig. 4.

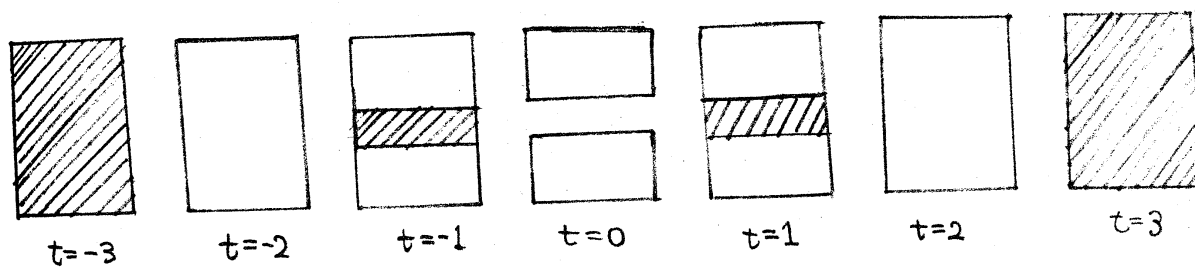


Fig. 4

1.4. Proof of Theorem 1.2. It suffices to prove Theorem 1.2 for the case  $n \geq 1$ . Assume  $F_n$  is unknotted. By definition,  $F_n$  bounds

a solid torus  $T_n$  of genus  $n$ . Let a system  $\{B_1, \dots, B_n\}$  be mutually disjoint  $n$  3-cells in  $T_n$ , obtained by thickening a system of meridional disks of  $T_n$ , such that  $B = \text{cl}(T_n - B_1 \cup \dots \cup B_n)$  is a 3-cell.  $B$  is ambient isotopic to a 3-cell in  $R^3[0]$ ; so we assume that  $B$  is contained in  $R^3[0]$ . Let  $L_n$  be a one-point-union of  $n$  1-spheres at  $v$  in  $\text{Int}(T_n)$  which is a spine of  $T_n$ , i.e., to which  $T_n$  collapses. Choose a sufficiently small, compact and connected neighborhood  $U(v)$  of  $v$  in  $L_n$  so that  $U(v)$  contains no vertices of  $L_n$  except for  $v$ . We may consider that  $U(v) = L_n \cup B$  and  $BX[-1 \leq t \leq 1] \cap (L_n - U(v)) = \emptyset$ . It is not hard to see that  $L_n$  is ambient isotopic to an  $n$ -leafed rose in  $R^3[0]$  by an ambient isotopy of  $R^4$  keeping  $BX[-1 \leq t \leq 1]$  fixed. So, we regard  $L_n$  as an  $n$ -leafed rose in  $R^3[0]$ . Let  $R_0^4 = \text{cl}(R^4 - BX[-1 \leq t \leq 1])$  and  $\text{cl}(L_n - U(v)) = \mathcal{L}_1 \cup \dots \cup \mathcal{L}_n$ , where  $\mathcal{L}_i$  are connected components. Note that  $\text{cl}(T_n - B) = B_1 \cup \dots \cup B_n$ . Now we shall show that there exist mutually disjoint regular neighborhoods  $H_i$  of  $\mathcal{L}_i$  in  $R_0^4$  that meet the boundary  $\partial R_0^4$  regularly and such that the pairs  $(B_i \subset H_i)$  are proper, i.e.,  $\partial B_i = (\partial H_i) \cup B_i$ . To show this, triangulate  $R_0^4$  so that  $B_1 \cup \dots \cup B_n$  is a subcomplex of  $R_0^4$  and so that  $\mathcal{L}_1 \cup \dots \cup \mathcal{L}_n$  is a subcomplex of  $B_1 \cup \dots \cup B_n$ . Let  $X$  and  $H'$  be the barycentric second derived neighborhoods of  $\mathcal{L}_1 \cup \dots \cup \mathcal{L}_n$  in  $B_1 \cup \dots \cup B_n$  and in  $R_0^4$ , respectively. It is easily seen that the pair  $(X \subset H')$  is proper. Since  $\text{cl}(B_1 \cup \dots \cup B_n - X)$  is homeomorphic to  $\text{cl}(F_n - \partial B)X[0, 1]$ ,  $B_1 \cup \dots \cup B_n$  is ambient isotopic to  $X$  by an ambient isotopy of  $R_0^4$ . Using this ambient isotopy, the desired pair  $(B_1 \cup \dots \cup B_n \subset H_1 \cup \dots \cup H_n)$  is obtained.

Next, by using the uniqueness theorem of regular neighborhoods, we may assume that  $H_i = N(\mathcal{L}_i, R_0^3) \times [-1 \leq t \leq 1]$ ,  $i = 1, 2, \dots, n$ , where  $R_0^3 = \text{cl}(R^3[0] - B)$  and  $N(\mathcal{L}_i, R_0^3)$  is a regular neighborhood of  $\mathcal{L}_i$  in  $R_0^3$  meeting the boundary  $\partial R_0^3$  regularly. More precisely, we can assume that  $\partial R_0^3 \cap N(\mathcal{L}_i, R_0^3) = (\partial B) \cap B_i$ .

Now we need the following lemma:

1.5. Lemma. Let a 1-sphere  $S^1$  be contained in a 2-sphere  $S^2$  and consider a proper surface  $Y$  in  $S^2 \times [0, 1]$ , (abstractly) homeomorphic to  $S^1 \times [0, 1]$ . If  $Y \cap S^2 \times 0 = S^1 \times 0$  and  $Y \cap S^2 \times 1 = S^1 \times 1$ , then  $Y$  is ambient isotopic to  $S^1 \times [0, 1]$  by an ambient isotopy of  $S^2 \times [0, 1]$  keeping  $S^2 \times 0$  and  $S^2 \times 1$  fixed.

By using Lemma 1.5,  $\text{cl}(\partial B_i - \partial B)$  is ambient isotopic to  $\text{cl}(\partial N(\mathcal{L}_i; R_0^3) - \partial B)$  by an ambient isotopy of  $\text{cl}(\partial H_i - \partial B \times [-1 \leq t \leq 1])$  keeping the boundary fixed. Hence by using a collar neighborhood of  $\text{cl}(\partial H_i - \partial B \times [-1 \leq t \leq 1])$  in  $R_0^4$ , we obtain that by an ambient isotopy of  $R_0^4$  keeping  $\partial R_0^4$  fixed. This implies that  $F_n$  is ambient isotopic to the boundary of a regular neighborhood of  $L_n$  in  $R^3[0]$ . Since the converse is obvious, the proof is completed.

1.6. Proof of Lemma 1.5. Let  $D \subset S^2$  be a 2-cell with  $\partial D = S^1$ . The 2-sphere  $Y \cup D \times 0 \cup D \times 1$  bounds the 3-cell  $C$  in  $S^2 \times [0, 1]$ , since  $S^2 \times [0, 1] \subset S^3$ . Let  $p \in \text{Int}(D)$  and choose a proper simple arc  $\alpha$  in  $C$  to which  $C$  collapses and such that  $\alpha \cap S^2 \times 0 = p \times 0$  and  $\alpha \cap S^2 \times 1 = p \times 1$ . Since there is an ambient isotopy of  $S^2 \times [0, 1]$  keeping  $S^2 \times 0$  and  $S^2 \times 1$  fixed and carrying  $\alpha$  to  $p \times [0, 1]$ , it follows from the uniqueness

theorem of regular neighborhoods that  $C$  is ambient isotopic to  $D \times [0,1]$  by an ambient isotopy of  $S^2 \times [0,1]$  keeping  $S^2 \times 0$  and  $S^2 \times 1$  fixed. This proves Lemma 1.5.

As one consequence of Theorem 1.2, we have the following corollary:

1.7. Corollary. For any unknotted surface  $F_n$  in  $R^4$ , the bounding solid torus  $T_n$  is unique up to ambient isotopies of  $R^4$ .

Proof. Let  $T_n$  be a solid torus in  $R^4$  with  $\partial T_n = F_n$ . It suffices to construct an ambient isotopy  $\{h_t\}$  of  $R^4$  such that  $h_1(T_n)$  is a regular neighborhood of an  $n$ -leafed rose in  $R^3[0]$ . By Theorem 1.2, we can assume that  $F_n$  is the boundary of a regular neighborhood of an  $n$ -leafed rose in  $R^3[0]$ . Let  $N(F_n)$  be a sufficiently thin regular neighborhood of  $F_n$  in  $R^3[0]$ . Then we may consider that the union of  $T_n$  and one component  $C(F_n)$  of  $N(F_n) - F_n$  is a solid torus  $T'_n$ . Since  $C(F_n)$  is homeomorphic to  $F_n \times (0,1]$ ,  $T'_n$  is ambient isotopic to  $T_n$ . Let  $T''_n$  be a regular neighborhood of an  $n$ -leafed rose in  $C(F_n)$  such that  $\text{cl}(T'_n - T''_n)$  is homeomorphic to  $F_n \times [0,1]$ . Since  $T'_n$  is ambient isotopic to  $T''_n$ , we complete the proof.

1.8. Note. It should be noted that for  $n \geq 1$  the bounding solid torus  $T_n$  is not unique up to ambient isotopies of  $R^4$  keeping  $F_n$  setwise fixed. Consider, for example, an unknotted surface  $F_1$  of genus 1 as in Fig.5.



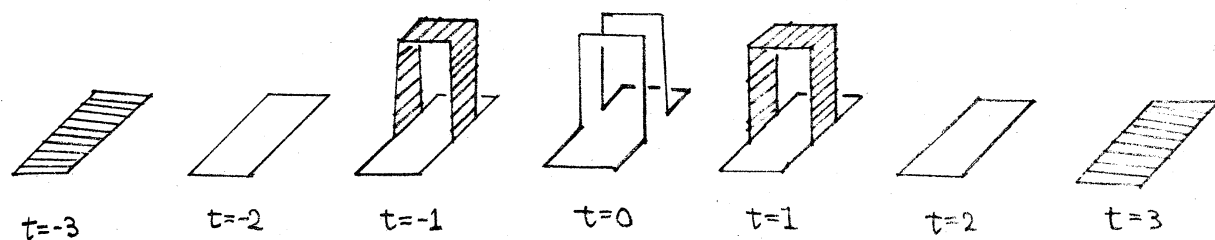


Fig. 5

This surface  $F_1$  bounds two kinds of solid tori  $T_1, T'_1$  as shown in Fig. 6.

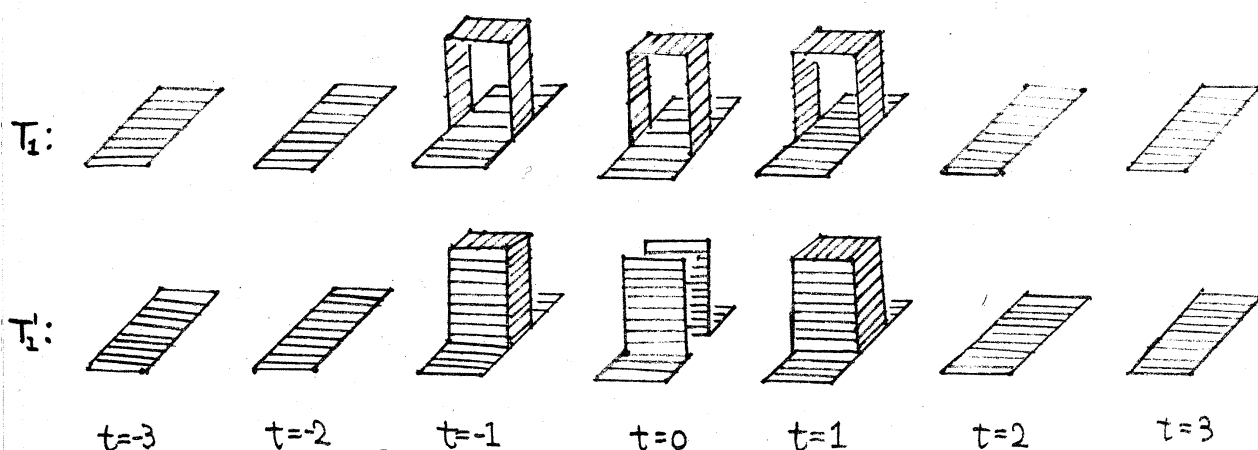


Fig. 6

Since the meridian curve of  $T_1$  relating critical bands of  $F_1$  is not a meridian curve of  $T'_1$ ,  $T_1$  is not ambient isotopic to  $T'_1$  by an ambient isotopy of  $R^4$  keeping  $F_1$  setwise fixed.

1.9. Note. Let  $F_n$  be unknotted in  $R^4$ . Consider the homeotopy

group  $\mathcal{H}(R^4, F_n)$  of auto-homeomorphisms of the pair  $(R^4, F_n)$  modulo the homeomorphisms ambient isotopic to the identity. By Theorem 1.2, the homeotopy group  $\mathcal{H}(R^4, F_n)$  is isomorphic to a homeotopy group  $\mathcal{H}(R^4, \partial T_n)$ , where  $\partial T_n$  is the boundary of a regular neighborhood  $T_n$  of an  $n$ -leafed rose in  $R^3[0]$ . So, we assume  $F_n = \partial T_n$ . Note 1.8 asserts that the group  $\mathcal{H}(R^4, F_n)$  is non-trivial. Let  $a_1, \dots, a_n$ ;  $b_1, \dots, b_n$  be the standard meridian and longitude curves of  $T_n \subset R^3[0]$ . The homeotopy group  $\mathcal{H}(R^4, F_n)$  contains the elements represented by the following auto-homeomorphisms;  $h_{(i_1, \dots, i_n)}$  and  $h^{(j)}$  such that

$$\begin{aligned} h_{(i_1, \dots, i_n)}(a_k) &= a_{i_k} \\ h_{(i_1, \dots, i_n)}(b_k) &= b_{i_k}, \end{aligned}$$

where  $(i_1, \dots, i_n)$  is a permutation on  $\{1, \dots, n\}$  defined by

$$(i_1, \dots, i_n) = \begin{pmatrix} 1 & \dots & n \\ i_1 & \dots & i_n \end{pmatrix}, \text{ and}$$

$$h^{(j)}(a_j) = b_j \quad h^{(j)}(a_k) = a_k$$

$$h^{(j)}(b_j) = a_j \quad h^{(j)}(b_k) = b_k, \quad k \neq j,$$

since  $T_n$  is contained in a 3-sphere in  $R^4$ . (Discussions on the orientation are now omitted.) Details of the homeotopy group

$\mathcal{H}(R^4, F_n)$  remain as an open problem. For example, is  $\mathcal{H}(R^4, F_n)$  isomorphic to the homeotopy group  $\mathcal{H}(F_n)$  of the surface  $F_n$ ?

## 2. Hyperboloidal Transformations

Let  $F$  be a (possibly non-connected) closed and oriented

surface in  $R^4$ . An oriented 3-cell  $B$  in  $R^4$  is said to span  $F$  as a 1-handle, if  $B \cap F = (\partial B) \cap F$  and this intersection is the union of disjoint two 2-cells, and if the surface  $cl[F \cup \partial B - (\partial B) \cap F]$  can have an orientation compatible with both the orientations of  $F - (\partial B) \cap F$  (induced from  $F$ ) and  $\partial B - (\partial B) \cap F$  (induced from  $B$ ). Also, an oriented 3-cell  $B$  in  $R^4$  spans  $F$  as a 2-handle, if  $B \cap F = (\partial B) \cap F$  and this intersection is homeomorphic to the annulus  $S^1 \times [0,1]$ , and if the surface  $cl[F \cup \partial B - (\partial B) \cap F]$  can have an orientation compatible with both the orientations of  $F - (\partial B) \cap F$  and  $\partial B - (\partial B) \cap F$ .

2.1. Definition. If  $B_1, \dots, B_m$  are mutually disjoint oriented 3-cells in  $R^4$  which span  $F$  as 1-handles, then the resulting oriented surface  $h^1(F; B_1, \dots, B_m) = cl[F \cup \partial B_1 \cup \dots \cup \partial B_m - F \cap (\partial B_1 \cup \dots \cup \partial B_m)]$  with orientation induced from  $F - F \cap (B_1 \cup \dots \cup B_m)$  is called the surface obtained from  $F$  by the hyperboloidal transformations along 1-handles  $B_1, \dots, B_m$ . Likewise, if  $B_1, \dots, B_m$  span  $F$  as 2-handles, the resulting oriented surface  $h^2(F; B_1, \dots, B_m) = cl[F \cup \partial B_1 \cup \dots \cup \partial B_m - F \cap (\partial B_1 \cup \dots \cup \partial B_m)]$  is called the surface obtained from  $F$  by the hyperboloidal transformations along 2-handles  $B_1, \dots, B_m$ .

We may have the following:

2.2. For arbitrary integers  $m$  and  $n$  with  $1 \leq m \leq n$ , if  $F_n$  is unknotted in  $R^4$ , then there exist mutually disjoint  $m$  3-cells  $B_1, \dots, B_m$  in  $R^4$  which span  $F_n$  as 2-handles and such that  $h^2(F_n; B_1, \dots, B_m)$  is an unknotted surface of genus  $n-m$ .

We shall show the following theorem which was partially suggested to the authors by T.Yajima:

2.3. Theorem. For arbitrary integers  $m$  and  $n$  with  $1 \leq m \leq n$ ,  
if  $F_n$  is unknotted in  $R^4$ , then one can find mutually disjoint  $m$   
3-cells  $B_1, \dots, B_m$  in  $R^4$  which span  $F_n$  as 2-handles and such  
that  $h^2(F_n; B_1, \dots, B_m)$  is a knotted surface of genus  $n-m$ . Further,  
every knotted surface in  $R^4$  is ambient isotopic to a surface  
 $h^2(F_n; B_1, \dots, B_m)$  with an unknotted surface  $F_n$  and spanning  
2-handles  $B_1, \dots, B_m$  for some  $m$  and  $n$  ( $m \leq n$ ).

The proof will be given later.

Combined 2.2 with Theorem 2.3, we conclude that the knot type  
of the surface  $h^2(F_n; B_1, \dots, B_m)$  in  $R^4$  depends on the choice of  
 $B_1, \dots, B_m$ , even if  $F_n$  is unknotted. (In case  $F_n$  is knotted, the  
 assertion has already known by T.Yajima[7].)

On the other hand, concerning 1-handles, we shall obtain the  
 following:

2.4. Theorem. Given an unknotted surface  $F_n$  and mutually  
disjoint  $m$  3-cells  $B_1, \dots, B_m$  in  $R^4$  which span  $F_n$  as 1-handles,  
then the resulting surface  $h^1(F_n; B_1, \dots, B_m)$  of genus  $n+m$  is  
necessarily unknotted.

2.5. Note. In case  $F_n$  is a knotted surface, then the knot  
 type of the surface  $h^1(F_n; B_1, \dots, B_m)$  depends on the choice of  
 $B_1, \dots, B_m$ . For example, let us consider the 2-sphere  $S$  illustrated  
 in Fig. 7.

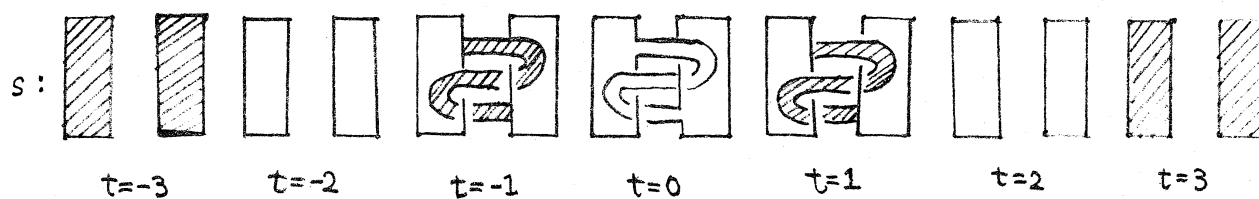


Fig. 7

This 2-sphere  $S$  is certainly knotted, since the fundamental group  $\pi_1(R^4-S)$  has a presentation  $(a,b: aba=bab)$  whose Alexander polynomial is  $t^2-t+1$ . [In fact, this 2-sphere has the same knot type as the spun 2-knot of a trefoil.] Let  $B, B'$  be two 3-cells that span  $S$  as 1-handles, as shown in Fig.8.

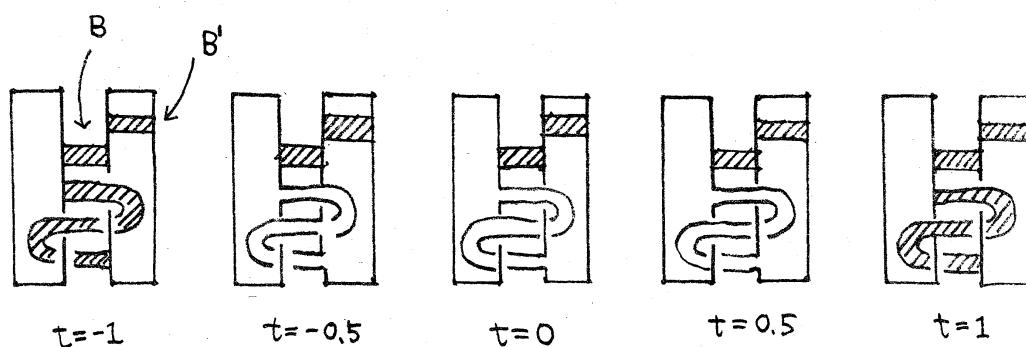


Fig. 8

The surfaces  $F_1 = h^1(S; B)$  and  $F'_1 = h^1(S; B')$  of genera 1 related to the 3-cells  $B$  and  $B'$  are illustrated in Fig. 9.

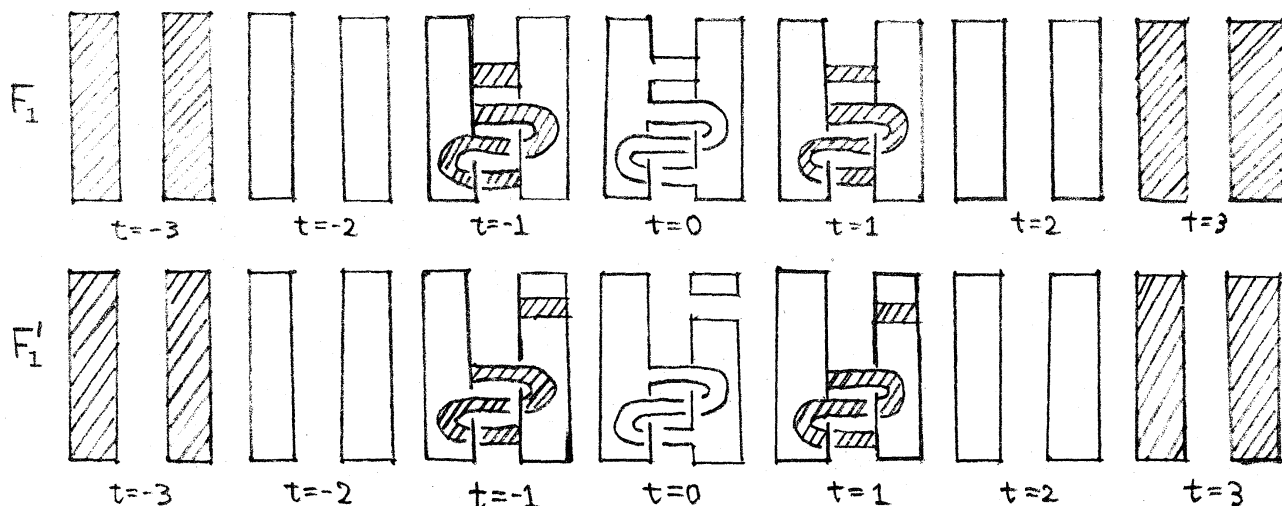


Fig. 9

It is easily seen that the fundamental group  $\pi_1(R^4 - F_1)$  is an infinite cyclic group [ In 2.7 we shall show that this  $F_1$  is actually unknotted.] and the fundamental group  $\pi_1(R^4 - F'_1)$  is isomorphic to the fundamental group  $\pi_1(R^4 - S)$  that is non-abelian. Hence the knot types of  $F_1$  and  $F'_1$  are distinct.

2.6. Proof of Theorem 2.4. We shall show the existence of a solid torus  $T_n$  of genus  $n$  in  $R^4$  with  $\partial T_n = F_n$  and  $\text{Int}(T_n) \cap B_i = \emptyset$ ,  $i = 1, 2, \dots, m$ . Then the desired result follows, since  $T_n \cup B_1 \cup \dots \cup B_m$  is a solid torus of genus  $n+m$  and since  $h^1(F_n; B_1, \dots, B_m) = \chi(T_n \cup B_1 \cup \dots \cup B_m)$ . Choose for each  $i$ ,  $i = 1, 2, \dots, m$ , a simple proper arc  $\alpha_i$  in  $B_i$  so that the union  $F_n \cup \alpha_1 \cup \dots \cup \alpha_m$  is a spine of the union

$F_n \cup B_1 \cup \dots \cup B_m$ . Since  $F_n$  is unknotted, we may consider  $F_n$  as the surface of genus  $n$  illustrated in Fig. 10.

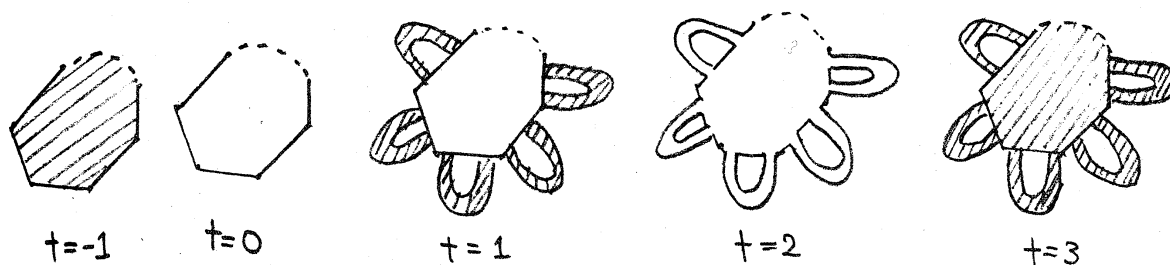


Fig. 10

By sliding  $B_1, \dots, B_m$  along  $F_n$  and by deforming  $B_1, \dots, B_m$  themselves, we can assume that  $\alpha_1, \dots, \alpha_m$  are attached to the circle in the level  $t = 0$ , i.e.,  $F_n \cap \mathbb{R}^3[0]$  in well order and that for each  $i$  the two attaching points of  $\alpha_i$  to  $F_n \cap \mathbb{R}^3[0]$  have compact and connected neighborhoods  $n_i^+$  and  $n_i^-$  in  $\alpha_i$  which are contained in the level  $t = 0$ . For  $m = 3$  we illustrated the situation in Fig. 11.

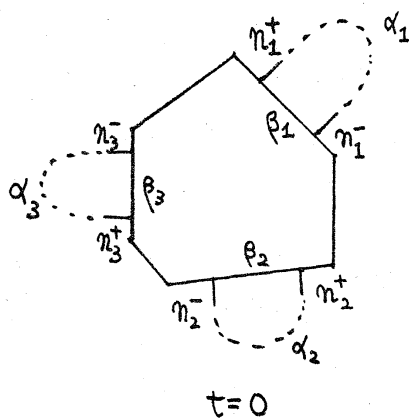


Fig. 11

For each  $i$ , let  $\beta_i$  be the part of  $F_n \cap \mathbb{R}^3[0]$  divided by  $\alpha_i$  as in Fig. 12. (For  $m = 1$  let  $\beta_1$  be any one of the two components of  $F_n \cap \mathbb{R}^3[0]$  divided by  $\alpha_1$ .) Further, for each  $i$ , let  $\alpha'_i = \text{cl}(\alpha_i - n_i^+ \cup n_i^-)$ . Now we join, for each  $i$ , the end points of  $\alpha'_i$  with a simple arc  $\gamma_i$  such that the loop  $\beta_i \cup n_i^+ \cup n_i^- \cup \gamma_i$  bounds a non-singular disk  $D_i$  in  $\mathbb{R}^3[0]$  with  $\text{Int}(D_i) \cap (F_n \cup \alpha_1 \cup \dots \cup \alpha_m) = \emptyset$ , as in Fig. 12.

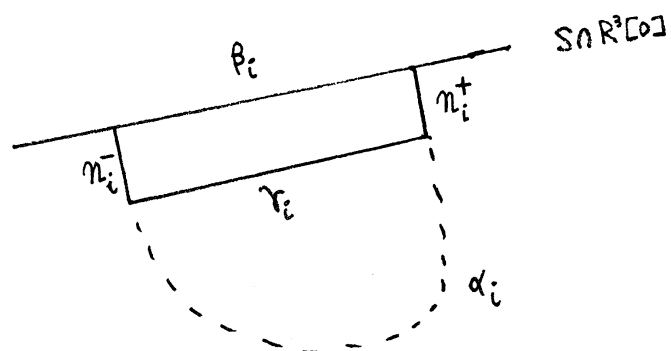


Fig. 12

The simple closed curve  $\gamma_i \cup \alpha'_i$  is in general not homologous to 0 in  $\mathbb{R}^4 - F_n$ . However, by twisting  $\gamma_i$  along the circle  $F_n \cap \mathbb{R}^3[0]$  (See for example Fig. 13.), we can assume that the simple closed curve  $\gamma_i \cup \alpha'_i$  is homologous to 0 in  $\mathbb{R}^4 - F_n$ .

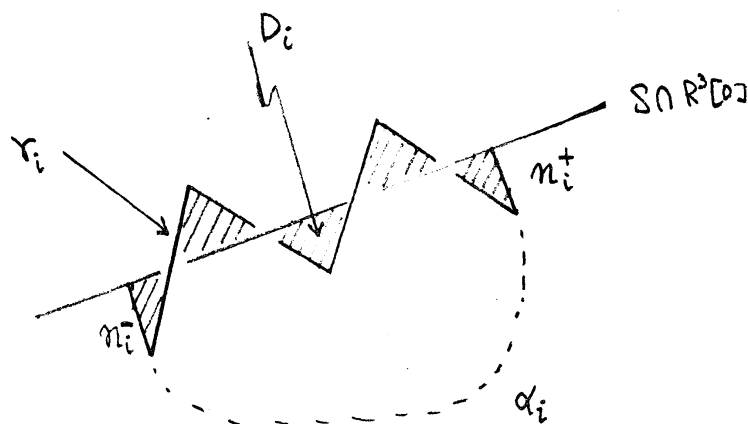


Fig. 13



Since  $F_n$  is unknotted, we have the Hurewicz isomorphism  $\pi_1(R^4 - F_n) \approx H_1(R^4 - F_n; \mathbb{Z})$ . Hence  $\gamma_i \cup \alpha'_i$  is null-homotopic in  $R^4 - F_n$ . By general position and by slight modifications,  $\gamma_i \cup \alpha'_i$ ,  $i = 1, 2, \dots, m$ , bound mutually disjoint non-singular disks  $d_i$  in  $R^4 - F_n$ . Thus,  $F_n \cup \alpha_1 \cup \dots \cup \alpha_m$  is ambient isotopic to  $F_n \cup (n_1^+ \cup \gamma_1 \cup n_1^-) \cup \dots \cup (n_m^+ \cup \gamma_m \cup n_m^-)$ . Hence  $F_n \cup \alpha_1 \cup \dots \cup \alpha_m$  is ambient isotopic to the standard surface of genus  $n$  with  $m$  attaching curves, as in Fig. 14.

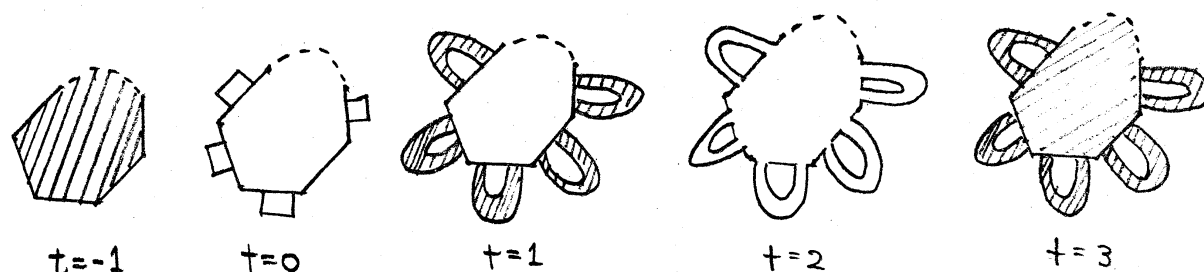


Fig. 14

Now by using the uniqueness theorem of regular neighborhoods, one can easily find a solid torus  $T_n$  of genus  $n$  in  $R^4$  with  $\partial T_n = F_n$  and  $\text{Int}(T_n) \cap B_i = \emptyset$ ,  $i = 1, 2, \dots, m$ . This completes the proof.

**2.7. Proof of Theorem 2.3.** We shall show that, for an unknotted surface  $F_1$  of genus 1, there exists a 3-cell  $B_1$  in  $R^4$  which spans  $F_1$  as a 2-handle and such that  $h^2(F_1; B_1)$  is a knotted 2-sphere. Then it is easy to find mutually disjoint 3-cells  $B_1, \dots, B_m$  which span an unknotted surface  $F_n$  as 2-handles and such

that  $h^2(F_n; B_1, \dots, B_m)$  is a knotted surface of genus  $n-m$  for arbitrary given  $m \leq n$ . We consider the surface  $F_1$  in Fig. 9. This surface is actually unknotted. In fact, let  $\bar{B}$  be the 3-cell which spans  $F_1$  as a 2-handle, illustrated in Fig. 15.

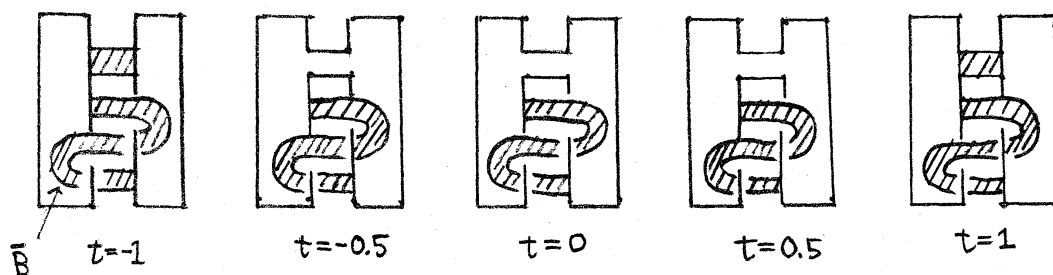


Fig. 15

The resulting 2-sphere  $S_0 = h^2(F_1; \bar{B})$  is clearly unknotted. Then Theorem 2.4 shows that the surface  $F_1 = h^1(S_0; \bar{B})$  is unknotted. Consider the 3-cell  $B$  in Fig. 8 that spans  $F_1$  as a 2-handle. The resulting 2-sphere  $h^2(F_1; B)$  is knotted, because  $h^2(F_1; B)$  is  $S$  in Fig. 7.

Secondly, we shall show that any knotted surface  $F$  in  $R^4$  is ambient isotopic to a surface  $h^2(F_n; B_1, \dots, B_m)$  with an unknotted surface  $F_n$  and spanning 2-handles  $B_1, \dots, B_m$  for some  $m$  and  $n$  ( $m \leq n$ ). Consider a compact, connected 3-manifold  $M$  in  $R^4$  with  $\partial M = F$ . It is not difficult to find mutually disjoint 3-cells  $B_1, \dots, B_m$  in  $M$  which span  $F$  as 1-handles and such that  $T = \text{cl}(M - B_1 \cup \dots \cup B_m)$  is a solid torus. [In fact, take a 2-complex  $K$  that is a spine of  $M$

and let  $K^{(1)}$  be the 1-skelton of  $K$ . Take the regular neighborhood  $T' = N(K^{(1)}; M)$  of  $K^{(1)}$  in  $M$ . We may consider that  $\text{cl}(K-T')$  consists of  $m$  2-cells  $\Delta_1, \Delta_2, \dots, \Delta_m$  for some  $m$ . For each  $i$ , let  $B'_i$  be a 3-cell thickening  $\Delta_i$  in  $\text{cl}(M-T')$ . The union  $M' = T' \cup B'_1 \cup \dots \cup B'_m$  is a regular neighborhood of  $K$  in  $M$ . Using the uniqueness theorem of regular neighborhoods, we obtain that  $M'$  is homeomorphic to  $M$ . Divide  $M$  into a solid torus  $T$  and 3-cells  $B_1, \dots, B_m$  corresponding to  $T'$  and  $B'_1, \dots, B'_m$ , respectively, utilizing this homeomorphism. The result follows.] Let  $F_n = \partial T$ , where  $n$  is the genus of  $T$ . By definition,  $F_n$  is unknotted. From construction, we have  $F = h^2(F_n; B_1, \dots, B_m)$ . This completes the proof.

A basic unsolved problem still remains that asks whether, given a knotted surface  $F_n$  of genus  $n$ ,  $n \geq 1$ , one can always find mutually disjoint  $n$  3-cells  $B_1, \dots, B_n$  in  $R^4$  which spans  $F_n$  as 2-handles and such that  $h^2(F_n; B_1, \dots, B_n)$  is a 2-sphere. (The resulting 2-sphere will be necessarily knotted by Theorem 2.4.)

The following shows that there is a knotted surface from which one can never produce a 2-sphere by the hyperbolic transformation along 2-handle without changing the fundamental groups:

2.8. Theorem. There exists a knotted surface  $F_n$  in  $R^4$  (for each  $n \geq 1$ ) such that

- (1) One can find mutually disjoint  $n$  3-cells  $B_1^0, \dots, B_n^0$  with  $h^2(F_n; B_1^0, \dots, B_n^0)$  a 2-sphere,
- (2)  $\pi_1(R^4 - F_n)$  is not isomorphic to  $\pi_1(R^4 - h^2(F_n; B_1, \dots, B_n))$  for any mutually disjoint  $n$  3-cells  $B_1, \dots, B_n$  with  $h^2(F_n; B_1, \dots, B_n)$  a 2-sphere.

Proof. It suffices to prove for the case  $n = 1$ . We shall show that the surface of genus 1 described in Fig. 16 is such a surface.

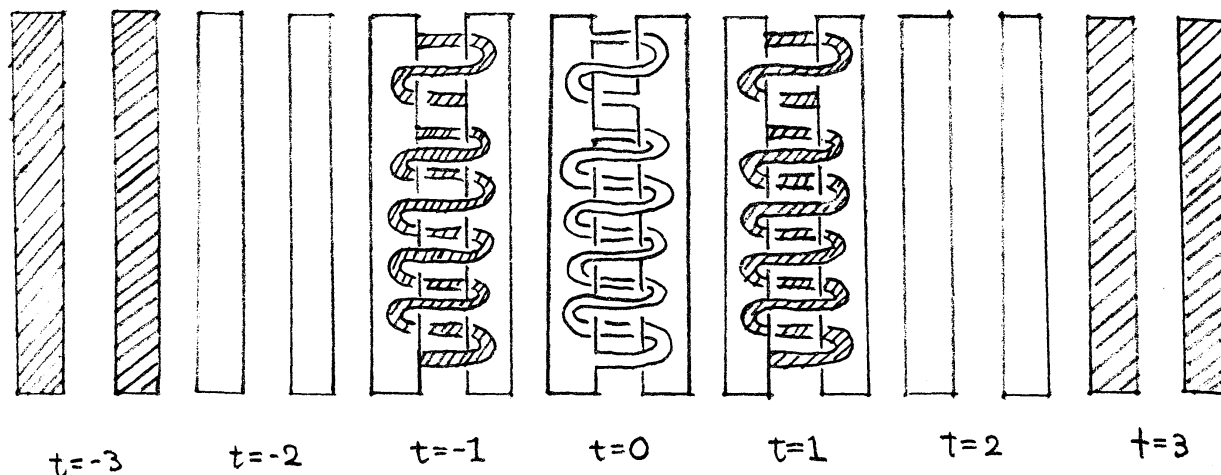


Fig. 16

This surface certainly satisfies (1). To see that it also satisfies (2), consider the fundamental group  $\pi$  of the complement of this surface in  $R^4$ .  $\pi$  has a presentation  $(a, b \mid ab = ba^2, ba^5 = a^5b)$ . (See for example R.H.Fox[1] or T.Yajima[7] for a calculation.) Obviously,  $H_1(\pi; Z) \approx Z$  and, by sending  $b$  of this presentation to  $t$ , a generator of an infinite cyclic group, the abelianized commutator subgroup  $\pi'/\pi''$  of  $\pi$  is isomorphic to  $Z[t]/(1-2t, 5t-5)$  as  $Z[t]$ -modules, where  $(1-2t, 5t-5)$  denotes the ideal over  $Z[t]$

generated by the polynomials  $1-2t$  and  $5t-5$ . Using the identity  $5 = 5(2t-1) + 2(5-5t)$ ,  $\pi'/\pi''$  is, consequently, isomorphic to  $\mathbb{Z}_5[t]/(2t-1)$  as  $\mathbb{Z}[t]$ -modules. In particular,  $\pi'/\pi''$  is isomorphic to  $\mathbb{Z}_5$  as abelian groups.

Now we need the following theorem that seems essentially the same as a result of M.A.Gutiérrez[3] (, although our approach is different from his.):

2.9.Theorem. Let  $G$  be a finitely presented group with  $H_1(G; \mathbb{Z}) = \mathbb{Z}$  and such that  $G'/G''$  is a finitely generated torsion group. If  $G$  is isomorphic to  $\pi_1(R^4 - S)$  for some 2-sphere  $S^2$  in  $R^4$ , then for any finite field  $F$  the first polynomial invariant  $a(t)$  of  $(G'/G'') \otimes_{\mathbb{Z}} F$  as  $F[t]$ -modules is reciprocal:  $a(t) \doteq a(t^{-1})$  up to units of  $F[t]$ . (The first polynomial invariant  $a(t)$  is defined to be the product  $f_1(t)f_2(t)\dots f_r(t)$  for a cyclic decomposition  $(G'/G'') \otimes_{\mathbb{Z}} F \cong F[t]/(f_1(t)) \oplus F[t]/(f_2(t)) \oplus \dots \oplus F[t]/(f_r(t))$  as  $F[t]$ -modules.)

Note that  $2t-1$  is the first polynomial invariant of  $(\pi'/\pi'') \otimes_{\mathbb{Z}} \mathbb{Z}_5$ . Since  $2t-1$  is not reciprocal in  $\mathbb{Z}_5[t]$ , it follows from 2.9 that  $\pi$  is not the fundamental group of any 2-sphere in  $R^4$ . This is enough to show (2). This completes the proof.

2.10. Proof of Theorem 2.9. By assumption,  $G$  is isomorphic to the group  $\pi_1(S^4 - S)$  for some 2-sphere  $S$  in a 4-sphere  $S^4$ . Let  $N(S)$  be the regular neighborhood of  $S$  in  $S^4$  and  $M = \text{cl}(S^4 - N(S))$ . Note that  $\partial M$  is homeomorphic to  $S^1 \times S^2$ . Consider the infinite cyclic

cover  $\tilde{M}$  of  $M$  associated with the Hurewicz epimorphism  $\pi_1(M) \rightarrow H_1(M; \mathbb{Z})$ . Since  $H_1(\tilde{M}; \mathbb{Z}) \cong G'/G''$  is a finitely generated torsion group, it follows from A.Kawauchi[5, Theorem 2.3] that  $H_*(\tilde{M}; \mathbb{Z})$  is finitely generated as an abelian group and that there is a duality

$$\cap \mu: H^2(\tilde{M}; \mathbb{Z}) \cong H_1(\tilde{M}, \partial \tilde{M}; \mathbb{Z}).$$

By the universal coefficient theorem,  $H^1(\tilde{M}; F)$  is canonically isomorphic to the torsion product  $\text{Tor}[H^2(\tilde{M}; \mathbb{Z}), F]$ , for  $H^1(\tilde{M}; \mathbb{Z}) = 0$ . Since the inclusion map  $\tilde{M} \subset (\tilde{M}, \partial \tilde{M})$  induces an isomorphism  $H_1(\tilde{M}; \mathbb{Z}) \cong H_1(\tilde{M}, \partial \tilde{M}; \mathbb{Z})$  as  $\mathbb{Z}[t]$ -modules and  $H_1(\tilde{M}; \mathbb{Z})$  is a finitely generated torsion group and  $F$  is a finite field, we obtain the composite isomorphism

$$\begin{aligned} \bar{\Psi}(\mu): H^1(\tilde{M}; F) &\cong \text{Tor}[H^2(\tilde{M}; \mathbb{Z}), F] \cong \text{Tor}[H_1(\tilde{M}, \partial \tilde{M}; \mathbb{Z}), F] \\ &\cong \text{Tor}[H_1(\tilde{M}; \mathbb{Z}), F] \cong H_1(\tilde{M}; F). \end{aligned}$$

The identity  $(tu)\cap \mu = t^{-1}(u)\cap \mu$  for any  $u \in H^2(\tilde{M}; \mathbb{Z})$ , then, induces the following commutative square of isomorphisms:

$$\begin{array}{ccc} H^1(\tilde{M}; F) & \xrightarrow{\bar{\Psi}(\mu)} & H_1(\tilde{M}; F) \\ t \downarrow & & \downarrow t^{-1} \\ H^1(\tilde{M}; F) & \xrightarrow{\bar{\Psi}(\mu)} & H_1(\tilde{M}; F). \end{array}$$

Since  $H^1(\tilde{M}; F)$  and  $H_1(\tilde{M}; F)$  are isomorphic as  $F[t]$ -modules, the first polynomial invariant  $a(t)$  of  $H_1(\tilde{M}; F)$  must be reciprocal:  $a(t) \doteq a(t^{-1})$ . This completes the proof.

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